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R-matrix formulation of KP hierarchies and their gauge equivalence

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The Adler–Kostant–Symes R-bracket scheme is applied to the algebra of pseudodifferential operators to relate the three integrable hierarchies: KP and its two modifications, known as non-standard integrable models. All three hierarchies are shown to be equivalent and a connection is established in the form of a symplectic gauge transformation. This construction results in a new representation of the W-infinity algebras in terms of four boson fields.

1. Introduction

One of the important and still unsolved problems of two-dimensional physics is to describe consistently systems with infinitely many functional (field) degrees of freedom. Among such systems the largest attention was attained by the Kadomtsev–Petviashvili (KP) completely integrable hierarchy, which proved to be relevant for a variety of physical problems. A recent and intriguing development in this field is the appearance of integrable hierarchies, including KP, in the matrix models known to describe, at multicritical points, $c \leqslant 1$ matter systems coupled to $D=2$ quantum gravity [1]. In particular, the partition function of both discrete matrix-models [2,3] and of continuum $c \leqslant 1$ string field theory [4,5] is expressed in terms of a constrained τ-function of the KP hierarchy. The coupling constants in the matrix model partition function, corresponding to various possible deviations from the critical points, are the evolution parameters in the KP hierarchy.

The most essential feature of the integrable hierarchies with an infinite number of degrees of freedom, which prompts their connection to 2D conformal field theories, $c \leqslant 1$ strings and their matrix model counterparts, is their hamiltonian structure [6,7]. Thus, the Virasoro algebra provides the second hamiltonian structure of the Korteweg–de Vries (KdV) hierarchy. This is the algebra of constraints on the partition function of the one-

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matrix model [1]. In the same way the W_N algebras [8], or more precisely, their semiclassical analogue – the Gelfand–Dickey algebras [9], give the second hamiltonian structures of the generalized KdV hierarchies. The W_N are the algebras of constraints on the partition function in multi-matrix and Kontsevich matrix models [2,3]. Finally, the $W_{1+\infty}$ algebra [10], which is isomorphic to the Lie algebra of differential operators on the circle $\mathcal{DOP}(S^1)$ [11], yields the first hamiltonian structure of the KP hierarchy [12–14].

An important open problem of the matrix model formulation of 2D quantum gravity is how to describe the interpolation between two different vacua (one, characterized by (p, q) conformal matter, and another one, by (p', q'), $p' \neq p$, $q' \neq q$). It is known [15] that this cannot be solved in terms of the ordinary KP evolution parameters. A hope to solve this problem is to analyse all integrable hierarchies which are equivalent to KP. This is also an interesting mathematical problem by its own.

In this letter we would like to contribute to this program. The powerful Adler–Kostant–Symes (AKS) scheme for Lie-algebraic construction of integrable models (substantially improved and extended by Reyman and Semenov-Tian-Shansky) [16] is applied to the algebra of pseudo-differential operators on the circle $\Psi\mathcal{DO}(S^1)$ [17]. This scheme permits the treatment of the three different integrable KP-like hierarchies: ordinary KP and its two modifications, known as non-standard integrable models [13], on an equal footing. They correspond to the three possible splittings of the algebra $\Psi\mathcal{DO}(S^1)$ into a linear sum of two subalgebras. The main result in that all three hierarchies are proved to be "gauge" equivalent via a generalized Miura transformation. The (field-dependent) "gauge" transformations, which are explicitly constructed, belong to well-defined subgroups of the formal group of pseudo-differential operators – the abelian group of operators of multiplication by a function and the group of diffeomorphisms on the circle (Virasoro group), respectively. For the first of the modified KP hierarchies, this "gauge" equivalence was previously established [18] in the "semiclassical" limit of KP (known in non-linear hydrodynamics as the Benney integrable hierarchy [19]).

As an important by-product, the above "gauge" transformations together with the AKS R-bracket scheme provide new explicit realizations of $W_{1+\infty}$ algebras in terms of an unconventional set of four boson fields.

2. The Adler–Kostant–Symes scheme and applications to KP hierarchies

2.1. General scheme and R operators

It is well known that the Lie algebra methods allow for a unifying treatment of integrable systems [6]. One of the main purposes of this paper is to describe a relation between the Lax formulation of various KP-type systems defined below and the R-operator approach [20,21] to the integrable systems based on the AKS scheme [16]. We first recall the notion of integrability.

Complete integrability. Consider a hamiltonian system with n degrees of freedom possessing standard hamiltonian structure with a hamiltonian $H(p, q)$ and Poisson bracket $\{\ ,\ \}$. A hamiltonian system is called completely (or Liouville) integrable if it has n conserved quantities (integrals of motion) $I_k(p, q)$, $k = 1, ..., n$, which are in involution: $\{I_i, I_j\} = 0$. For such a system we can find the action-angle variables and write the general solution to the equations of motion.

Lax formulation. For infinite-dimensional integrable hamiltonian systems, there exists the convenient Lax (or "zero-curvature") formulation [6]. In the Lax formulation the dynamical equations of motion can be written in terms of a Lax pair L, P, with values in some Lie algebra $\mathcal{G}$, as the Lax-type equation

$$\frac{dL}{dt} = [L, P] . \tag{1}$$

The Lax formulation leads straightforwardly to the construction of the integrals of motion. Namely, for any Ad-invariant function I on $\mathcal{G}$, $I(L)$ is a constant of motion. In fact, it can be shown that any completely integrable hamiltonian system admits a Lax representation (at least locally) [22].

The KP hierarchy. An important example of integrable systems admitting the Lax formulation is given by the KP hierarchy consisting of the following family of Lax equations:

$$\frac{\partial L}{\partial t_r} = [L, L^r_+], \quad r=1, 2, 3, \ldots, \tag{2}$$

where L is a pseudo-differential operator

$$L = \mathrm{D} + \sum_{i=1}^{\infty} u_i \mathrm{D}^{-i}. \tag{3}$$

The subscript $(+)$ means taking the purely differential part of L^r and $t=\{t_r\}$ are the evolution parameters (infinitely many time coordinates). The flows (2) are bi-hamiltonian [7], i.e. there exist two Poisson bracket structures $\{\ ,\ \}_{1,2}$, such that we can rewrite (2) as

$$\frac{\partial L}{\partial t_r} = \{H_r, L\}_2 = \{H_{r+1}, L\}_1 . \tag{4}$$

Here the hamiltonians for the KP hierarchy are $H_r = r^{-1} \int \operatorname{Res} L^r$ (Res denotes the coefficient in front of D^{-1}). The second equality in (4) is a particular case of the so-called Lenard relations $\{H_{r+1}, L\}_m = \{H_r, L\}_{m+1}$ for a hierarchy of Poisson bracket structures $m=1, 2, \ldots$. There exists a fundamental theorem (see e.g. ref. [23]) connecting the notion of integrability with the property of possessing a bi-hamiltonian structure, which establishes the KP system as integrable.

The AKS scheme. A very wide class of integrable models can be constructed through the application of the AKS method having roots in the coadjoint orbit formulation.

Let G denote a Lie group and $\mathscr{G}$ be its Lie algebra. G acts on $\mathscr{G}$ by the adjoint action: $\mathrm{Ad}(g)X = gXg^{-1}$, with $g \in \mathrm{G}$ and $X \in \mathscr{G}$. Let $\mathscr{G}^*$ be the dual space of $\mathscr{G}$ relative to a non-degenerate bilinear form $\langle\ |\ \rangle$ on $\mathscr{G}^* \times \mathscr{G}$. The corresponding coadjoint action of $\mathscr{G}$ on $\mathscr{G}^*$ is obtained from the duality of $\langle\ |\ \rangle$: $\langle \mathrm{Ad}^*(g) U | X \rangle = \langle U | \mathrm{Ad}(g^{-1}) X \rangle$. We will denote the infinitesimal versions of adjoint and coadjoint transformations by $\mathrm{ad}(Y)$ and $\mathrm{ad}^*(Y)$ (for $g = \exp(Y)$).

There exists a natural Poisson structure on the space $\mathrm{C}^\infty(\mathscr{G}^*, \mathbb{R})$ of smooth, real-valued functions on $\mathscr{G}^*$, called the Lie–Poisson (LP) bracket. The LP bracket for $F, H \in \mathrm{C}^\infty(\mathscr{G}^*, \mathbb{R})$ is given by

$$\{F, H\}(U) = -\langle U | [\nabla F(U), \nabla H(U)] \rangle , \tag{5}$$

where the gradient ∇F: $\mathscr{G}^* \to \mathscr{G}$ is defined by the standard formula $(\mathrm{d}/\mathrm{d}t) F(U+tV)|_{t=0} = \langle V | \nabla F(U) \rangle$ and where $[\ ,\ ]$ is the standard Lie bracket on $\mathscr{G}$. It follows clearly that $\{\ ,\ \}$ is antisymmetric and it is also easy to verify the Jacobi identity. On each orbit in $\mathscr{G}^*$ the LP bracket gives rise to a non-degenerate symplectic structure. Moreover, for any hamiltonian H on such an orbit we have a hamiltonian equation $\mathrm{d}U/\mathrm{d}t = \mathrm{ad}^*(\nabla H(U))U$.

We now introduce the R-operator (generalized R-matrix) as a linear map from a Lie algebra $\mathscr{G}$ to itself such that the bracket

$$[X, Y]_R \equiv \tfrac{1}{2}[RX, Y] + \tfrac{1}{2}[X, RY] \tag{6}$$

defines a second Lie structure on $\mathscr{G}$ [20]. The modified Yang–Baxter equation (YBE) for the R-matrix must hold in order to ensure the Jacobi relation.

We can furthermore introduce a new LP bracket $\{\ ,\ \}_R$ called R-bracket by substituting the usual Lie bracket $[\ ,\ ]$ for the R-Lie bracket $[\ ,\ ]_R$ (6) in (5):

$$\{F, H\}_R(U) = -\langle U | [\nabla F(U), \nabla H(U)]_R \rangle . \tag{7}$$

A function H on $\mathscr{G}^*$ is called Ad^*-invariant (Casimir) if $H[\mathrm{Ad}^*(g)U] = H[U]$ or, infinitesimally, $\mathrm{ad}^*(\nabla H(U))(U) = 0$ for each $U \in \mathscr{G}^*$. Then one can show [20] that (1) the ad^*-invariant functions are in involution with respect to both brackets (5) and (7); (2) the hamiltonian equation on $\mathscr{G}^*$ takes the following (generalized Lax) form:

$$dU/dt = \tfrac{1}{2}\,\mathrm{ad}^*(R(\nabla H(U)))U\,, \tag{8}$$

corresponding to the equations of motion $dF/dt = \{H, F\}_R$ for $F \in C^\infty(\mathscr{G}^*, \mathbb{R})$.

Hence the above R-matrix technique leads to a direct construction of integrable systems based on Casimir functions on $\mathscr{G}^*$. The basic realization of this technique arises when the Lie algebra $\mathscr{G}$ decomposes as a vector space into two subalgebras $\mathscr{G}_+$ and $\mathscr{G}_-$, i.e. $\mathscr{G} = \mathscr{G}_+ \otimes \mathscr{G}_-$. Let $P_\pm$ be the corresponding projections on $\mathscr{G}_\pm$. Then $R = P_+ - P_-$ satisfies the modified YBE and provides a specific realization for the above scheme.

2.2. AKS construction of three KP hierarchies

Here we will illustrate the AKS construction on $\mathscr{G} = \Psi\mathscr{D}\mathscr{O}(S^1)$ – the Lie algebra of pseudo-differential operators on a circle. Recall that an arbitrary pseudo-differential operator $X(x, D_x) = \sum_{k \geqslant -\infty} X_k(x) D_x^k$ is conveniently represented by its symbol [17] – a Laurent series in the variable ξ:

$$X(\xi, x) = \sum_{k \geqslant -\infty} X_k(x)\xi^k\,, \tag{9}$$

and the operator multiplication corresponds to the following symbol multiplication:

$$X(\xi, x) \circ Y(\xi, x) = \sum_{N \geqslant 0} h^N \frac{1}{N!} \frac{\partial^N X}{\partial \xi^N} \frac{\partial^N Y}{\partial x^N}\,, \tag{10}$$

which determines a Lie algebra structure given by a commutator $[X, Y] \equiv (1/h)(X \circ Y - Y \circ X)$. Explicitly we have

$$[X(\xi, x), Y(\xi, x)] = \sum_{N \geqslant 1} (h)^{N-1} \frac{1}{N!} \left(\frac{\partial^N X}{\partial \xi^N} \frac{\partial^N Y}{\partial x^N} - \frac{\partial^N X}{\partial x^N} \frac{\partial^N Y}{\partial \xi^N} \right). \tag{11}$$

The constant h appearing in (10) and (11) has the meaning of a deformation parameter and will henceforth be taken as $h = 1$. The limit $h \to 0$ defines the semiclassical limit of $\Psi\mathscr{D}\mathscr{O}(S^1)$, where the Lie bracket (11) reduces to a two-dimensional Poisson bracket: $[X(\xi, x), Y(\xi, x)] = (\partial X/\partial\xi)(\partial Y/\partial x) - (\partial X/\partial x)(\partial Y/\partial \xi)$.

Using the Adler trace one next defines an invariant, non-degenerate bilinear form:

$$\langle L | X \rangle \equiv \mathrm{Tr}_A(LX) = \int dx\, \mathrm{Res}_\xi\, L(\xi, x) \circ X(\xi, x)\,, \tag{12}$$

which allows an identification of the dual space $\mathscr{G}^*$ with $\mathscr{G}$ and of the coadjoint action with the adjoint action.

There exist three natural decompositions of $\mathscr{G}$ into a linear sum of two subalgebras:

$$\mathscr{G}^l_+ = \left\{ X_+ \equiv X_{\geqslant l} = \sum_{i=l}^{\infty} X_i(x)\xi^i \right\}, \quad \mathscr{G}^l_- = \left\{ X_- \equiv X_{<l} = \sum_{i=-l+1}^{\infty} X_{-i}(x)\xi^{-i} \right\}, \tag{13}$$

labelled by the index l taking three values $l = 0, 1, 2$. For each l we clearly have $\mathscr{G} = \mathscr{G}^l_+ \otimes \mathscr{G}^l_-$. Correspondingly the dual spaces to subalgebras $\mathscr{G}^l_\pm$ are given by

$$\mathscr{G}^{l*}_+ = \left\{ L_- \equiv L_{<-l} = \sum_{i=l+1}^{\infty} \xi^{-i} \circ u_{-i}(x) \right\}, \quad \mathscr{G}^{l*}_- = \left\{ L_+ \equiv L_{\geqslant -l} = \sum_{i=-l}^{\infty} \xi^i \circ u_i(x) \right\}. \tag{14}$$

Note that in (14) the differential operators are put to the left.

Defining $R_l = P_+ - P_-$ for each of the three cases one finds that the R-bracket is given by

$$[X, Y]_{R_l} = [X_{\geqslant l}, Y_{\geqslant l}] - [X_{<l}, Y_{<l}] \ . \tag{15}$$

Furthermore, from the general relation for the R-coadjoint action of $\mathscr{G}$ on its dual space $\mathrm{ad}^*_R(X)L = \frac{1}{2}\,\mathrm{ad}^*(RX)L + \frac{1}{2}R^*\mathrm{ad}^*(X)L$ we find that the infinitesimal shift along an R-coadjoint orbit $\mathrm{O}(R_l)$ has the form

$$\delta_{R_l} L \equiv \mathrm{ad}^*_{R_l}(X)L = [\mathrm{ad}^*(X_+)L_-]_- - [\mathrm{ad}^*(X_-)L_+]_+ \equiv [X_{\geqslant l}, L_{<-l}]_{<-l} - [X_{<l}, L_{\geqslant -l}]_{\geqslant -l} \,. \tag{16}$$

Henceforth, the subscripts $\pm$ will denote projections on $\mathscr{G}_{l\pm}$ and $\mathscr{G}^*_{l\mp}$, as in (13), (14). Also, we shall skip the sign $\circ$ in symbol products for brevity.

We will now discuss in greater detail the hamiltonian structure of the integrable systems given by the three decompositions labelled by $l=0, 1, 2$ as defined by the AKS scheme with the hamiltonian equations of motion (8). We will call the resulting hierarchies the KP_l hierarchies.

- $\mathrm{KP}_{l=0}$: Here we take the R-coadjoint orbit of the form $\mathrm{O}(R_0) = \{L = \xi + L_- \equiv \xi + \sum_{k=1}^{\infty} \xi^{-k} u_k(x)\}$. Choosing as a Casimir the function $H_{m+1} = [1/(m+1)] \int \mathrm{d}x\, \mathrm{Res}_\xi L^{m+1}$ we get from (8)

$$\frac{\partial L}{\partial t_m} = \tfrac{1}{2}\,\mathrm{ad}^*((\nabla H_{m+1})_+ - (\nabla H_{m+1})_-)L = \mathrm{ad}^*((L^m)_+)L\,, \tag{17}$$

with $(L^m)_+ = \sum_{j\geqslant 0} (\delta H_{m+1}/\delta u_{j+1}(x))\xi^j$. We recognize in (17) the standard KP flow equation (2). The corresponding hamiltonian structure is found to be induced by the LP structure: $\{u_i(x), u_j(y)\}R_0 = \Omega^{(l=0)}_{i-1,j-1}(u(x))\delta(x-y)$, where the form on the RHS is given by [12,13]

$$\Omega^{(l)}_{i,j}(u(x)) = -\sum_{k=0}^{i+l} \binom{i+l}{k} u_{i+j+l-k+1}(x) \mathrm{D}_x^k + \sum_{k=0}^{j+l} (-1)^k \binom{j+l}{k} \mathrm{D}_x^k u_{i+j+l-k+1}(x) \tag{18}$$

for $l=0$. This LP bracket algebra is isomorphic to the centreless $\mathrm{W}_{1+\infty}$ algebra [14]. In conclusion we have found that $\mathrm{KP}_{l=0}$ is the standard KP hierarchy.

- $\mathrm{KP}_{l=1}$: Here we first consider elements of $\mathscr{G}^{1*}_-$ of the type $L_+ = \xi + u_0 + \xi^{-1}u_1$, which preserve their form under $\delta_{R_1} L_+ = \mathrm{ad}^*_{R_1}(X)L_+$, i.e. they span an R_1-orbit of finite functional dimension 2. Calculation of the Poisson bracket according to (7),

$$\{\langle L_+ | X\rangle, \langle L_+ | Y\rangle\}_{R_1} = -\langle L_+ | [X, Y]_{R_1}\rangle\ , \tag{19}$$

yields the R-brackets: $\{u_0(x), u_1(y)\}_{R_1} = -\delta'(x-y)$ and zero otherwise. We then define a complete Lax operator defined as $L^{(1)} = L_+ + L_- = \xi + u_0 + \xi^{-1}u_1 + \sum_{i\geqslant 2} \xi^{-i} v_{i-2}$. Application of (19) gives a hamiltonian structure that is a direct sum of the matrix $P^{(1)}$ associated with the modes $\{u_0, u_1\}$ and the hamiltonian structure $\Omega^{(1)}$ associated with $\{v_i | i \geqslant 0\}$ [13]:

$$\begin{pmatrix} P^{(1)} & 0 \\ 0 & \Omega^{(1)} \end{pmatrix}, \quad \text{with } P^{(1)} = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix}. \tag{20}$$

Note that $\Omega^{(1)}$ (18) corresponds to the centreless W_∞ algebra.

- $\mathrm{KP}_{l=2}$: Here elements of $\mathscr{G}^{2*}_-$ of the form $L_+ = \xi u_{-1} + u_0 + \xi^{-1}u_1 + \xi^{-2}u_2$, span an invariant subspace under $\delta_{R_2} L_+ = \mathrm{ad}^*_{R_2}(X)L_+$, i.e. they form an R_2-orbit of finite functional dimension 4. Defining the complete Lax operator $L^{(2)} = L_+ + L_- = \xi u_{-1} + u_0 + \xi^{-1}u_1 + \xi^{-2}u_2 + \sum_{i\geqslant 3} \xi^{-i} w_{i-3}$ we find from eq. (19) the corresponding hamiltonian structure to be [13]

$$\begin{pmatrix} P^{(2)} & 0 \\ 0 & \Omega^{(2)} \end{pmatrix}, \quad \text{with } P^{(2)} = \begin{pmatrix} 0 & 0 & 0 & -u_{-1}\mathrm{D} + u'_{-1} \\ 0 & 0 & -u_{-1}\mathrm{D} & u_{-1}\mathrm{D}^2 + u'_0 \\ 0 & -\mathrm{D}u_{-1} & 0 & \mathrm{D}u_1 \\ -\mathrm{D}u_{-1} - u'_{-1} & -\mathrm{D}^2 u_{-1} - u'_0 & u_1 \mathrm{D} & u_2\mathrm{D} + \mathrm{D}u_2 \end{pmatrix},$$

where $P^{(2)}$ and $\Omega^{(2)}$ (18) are associated with $\{u_{-1}, u_0, u_1, u_2\}$ and $\{w_i | i \geqslant 0\}$, respectively. The LP structure $P^{(2)}$

is easily recognized as a semidirect product of the (centreless) Virasoro algebra generated by the spin-2 field u_2 with a subalgebra generated by the conformal fields $\{u_{-1}, u_0+\partial_x u_{-1}, u_1\}$ with spins $-1, 0, 1$, respectively. Also, the LP structure with $\Omega^{(2)}$ (18) corresponds to the centreless algebra $W_\infty^{\geq 3}$ which is a subalgebra of W_∞ containing all generators of spin $\geqslant 3$.

3. "Gauge" equivalence of modified KP hierarchies to ordinary KP

3.1. Ordinary coadjoint action on R-coadjoint orbits as a generalized Miura transformation

In this section we shall explicitly construct symplectic (hamiltonian) maps among the various R-coadjoint orbits $O(R)$: Φ: $O(R)\to O(\tilde{R})$, where $R, \tilde{R}=R_0, R_1, R_2$. The term "symplectic" ("hamiltonian") means that under the map Φ, the LP bracket structure on $O(\tilde{R})$ is transformed into the LP bracket structure on $O(R)$:

$$\{\tilde{F}_1, \tilde{F}_2\}_{\tilde{R}}(\Phi(L)) = \{\tilde{F}_1(\Phi(L)), \tilde{F}_2(\Phi(L))\}_R , \tag{21}$$

where $\tilde{F}_{1,2}$ are arbitrary functions on $O(\tilde{R})$ and we used notations L and $\tilde{L}=\Phi(L)$ for the coordinates on $O(R)$ and $O(\tilde{R})$, respectively. As a consequence of (21), the infinite set of involutive integrals of motion $\{\tilde{H}_N[\tilde{L}]\}$ of the integrable system on $O(\tilde{R})$ are transformed into those of the integrable system on $O(R)$: $H_N[L]=\tilde{H}_N[\Phi(L)]$.

To this end we observe that the ordinary coadjoint actions of the Lie algebra $\mathcal{G}=\Psi\mathcal{DO}(S^1)$ and the group $G=\Psi DO(S^1)$ on the dual space $\mathcal{G}^*=\Psi\mathcal{DO}^*(S^1)$ do *not* commute with any of the R-coadjoint actions (16), i.e. $ad^*(\ )$ and $Ad^*(\ )$ intertwine the orbits for different R-coadjoint actions. Thus, it is natural to look for the map Φ:$O(R)\to O(\tilde{R})$ in the form

$$\tilde{L}\equiv\Phi(L)=Ad^*(g(L))L , \tag{22}$$

where the group element $g(L)\in\Psi DO(S^1)$ depends in general on the point L in $O(R)\subset\Psi\mathcal{DO}^*(S^1)$.

It is in the sense of eqs. (21) and (22) that the integrable systems on the orbits $O(R)$ for different R-matrices are called "gauge" equivalent. Also, from the point of view of eq. (21), i.e. mapping of one Poisson bracket structure of an integrable model into another one, the specific form of the $\Psi DO(S^1)$ group coadjoint action (22), mapping $O(R)$ into $O(\tilde{R})$, may be called a generalized Miura transformation.

Let us note the following important property of $Ad^*(g(L))$ in (22). It does not preserve the dual projections P^*_-, $\tilde{P}^*_-$ on $O(R)$ and $O(\tilde{R})$, respectively: $\tilde{P}^*_+ Ad^*(g(L))P^*_- \neq 0$.

It is sufficient to prove the "gauge" equivalence for linear functions on $O(\tilde{R})$, namely

$$\{\langle\tilde{L}|\tilde{X}\rangle, \langle\tilde{L}|\tilde{Y}\rangle\}_{\tilde{R}}|_{\tilde{L}=Ad^*(g(L))L} = \{\langle Ad^*(g(L))L|\tilde{X}\rangle, \langle Ad^*(g(L))L|\tilde{Y}\rangle\}_R . \tag{23}$$

Applying the general formula (6) to the RHS of (23) we have

$$\{\langle\Phi(L)|\tilde{X}\rangle, \langle\Phi(L)|\tilde{Y}\rangle\}_R = -\langle L|[\nabla_L\langle\Phi(L)|\tilde{X}\rangle, \nabla_L\langle\Phi(L)|\tilde{Y}\rangle]_R\rangle , \tag{24}$$

with

$$\nabla_L\langle\Phi(L)|\tilde{X}\rangle = [g^{-1}(L)\tilde{X}g(L)](\xi,x) - \int dy\, \mathrm{Res}_\zeta\left(\frac{\delta g(L)(\zeta,y)}{\delta L(\xi,x)} g^{-1}(L)(\zeta,y)[\tilde{X},\Phi(L)](\zeta,y)\right).$$

3.2. "Gauge" transformation of $KP_{l=1}$ to ordinary KP

Let us first specialize eq. (22) to the case Φ: $O(R)\equiv O(KP_{l=1})\to O(\tilde{R})\equiv O(KP_{l=0})$, i.e.

$$\tilde{L}\equiv\xi+\sum_{k=1}^{\infty}\xi^{-k}\tilde{u}_k(x) = Ad^*(g_0(L))\left(\xi+u_0(x)+\xi^{-1}u_1(x)+\sum_{k=2}^{\infty}\xi^{-k}v_{k-2}(x)\right). \tag{25}$$

The "gauge" subgroup is easily found to be the abelian group of multiplication operators,

$$g_0(L)=\exp[\phi_0(x)],\quad \partial_x\phi_0(x)=u_0(x)\,, \tag{26}$$

by using the simple formula

$$\exp[\phi_0(x)]\xi\exp[-\phi_0(x)]=\xi-\partial_x\phi_0(x)\,. \tag{27}$$

Furthermore, from the structure of $\tilde{L}$ (25), we find that only the (+) parts of $\tilde{X}$ and $\tilde{Y}$ contribute in (24), i.e. $\tilde{X}\equiv\tilde{X}_{\geqslant 0}\in\mathcal{G}_{+}^{l=0}$ (13). Finally, for $g(L)=g_0(L)$ (26), we have

$$\frac{\delta g_0(L)(\zeta,y)}{\delta L(\xi,x)}\,g_0^{-1}(L)(\zeta,y)=-\tfrac{1}{2}\epsilon(x-y)\xi^{-1}\,. \tag{28}$$

Now specializing eq. (24) by taking into account eqs. (25)–(28) and the form of the R-commutator (15) for $l=1$, we obtain for $\tilde{L}\equiv\exp(\phi_0)L\exp(-\phi_0)$

$$\begin{aligned}\{\langle\tilde{L}|\tilde{X}\rangle,\langle\tilde{L}|\tilde{Y}\rangle\}_{KP_{l=1}}=&-\langle\tilde{L}|\exp(\phi_0)\{[[\exp(-\phi_0)\tilde{X}\exp(\phi_0)]_{\geqslant 1},[\exp(-\phi_0)\tilde{Y}\exp(\phi_0)]_{\geqslant 1}]\\&-[A_1(\phi_0,\tilde{L})(\tilde{X}),A_1(\phi_0,\tilde{L})(\tilde{Y})]\}\exp(-\phi_0)\rangle\,,\end{aligned} \tag{29}$$

where $A_1(\phi_0,\tilde{L})(\tilde{X})\equiv[\exp(-\phi_0)\tilde{X}\exp(\phi_0)]_{(0)}+(\partial_x^{-1}\mathrm{Res}_\xi[\tilde{X},\tilde{L}])\xi^{-1}$ and the subscript (0) means taking the zeroth order part of the ξ-expansion of the corresponding symbol. Note that in any term on the RHS of (29) of the form $\langle\tilde{L}|\tilde{Z}\rangle$, only the projection $\tilde{Z}_{\geqslant -2}$ contributes. Using the simple identity

$$\begin{aligned}&\exp(\phi_0)[[\exp(-\phi_0)\tilde{X}\exp(\phi_0)]_{\geqslant 1},[\exp(-\phi_0)\tilde{Y}\exp(\phi_0)]_{\geqslant 1}]\exp(-\phi_0)\\&\quad=[\tilde{X},\tilde{Y}]-[[\exp(-\phi_0)\tilde{X}\exp(\phi_0)]_{(0)},\tilde{Y}]-[\tilde{X},[\exp(-\phi_0)\tilde{Y}\exp(\phi_0)]_{(0)}]\end{aligned} \tag{30}$$

to rewrite the first commutator on the RHS of (29), we easily find that the contribution of the terms in the second commutator on the RHS of (29) are precisely cancelled by the second and the third terms on the RHS of (30). Thus, (29) reduces to the form

$$\{\langle\tilde{L}(L)|\tilde{X}\rangle,\langle\tilde{L}(L)|\tilde{Y}\rangle\}_{\mathrm{KP}_{l=1}}=-\langle\tilde{L}|[\tilde{X},\tilde{Y}]\rangle=\{\langle\tilde{L}|\tilde{X}\rangle,\langle\tilde{L}|\tilde{Y}\rangle\}_{\mathrm{KP}}\,, \tag{31}$$

which establishes the "gauge" equivalence of $\mathrm{KP}_{l=1}$ and KP, i.e. that the generalized Miura-like transformation (25), (26) maps the Poisson bracket structure of KP into that of $\mathrm{KP}_{l=1}$ and vice versa.

3.3. "Gauge" transformation of $KP_{l=2}$ to ordinary KP

It is simpler to first establish the "gauge" equivalence between the modified KP hierarchies $\mathrm{KP}_{l=2}$ and $\mathrm{KP}_{l=1}$. The desired result follows by combining the results of this and the previous subsections.

Specializing eq. (22) to the case $\Phi{:}\,\mathrm{O}(R)\equiv\mathrm{O}(\mathrm{KP}_{l=2})\to\mathrm{O}(\tilde{R})\equiv O(\mathrm{KP}_{l=1})$, we have

$$\begin{aligned}\tilde{L}&\equiv\xi+\tilde{u}_0(x)+\xi^{-1}\tilde{u}_1(x)+\sum_{k=2}^{\infty}\xi^{-k}v_{k-2}(x)\\&=\mathrm{Ad}^*(g_1(L))\left(\xi u_{-1}(x)+u_0(x)+\xi^{-1}u_1(x)+\xi^{-2}u_2(x)+\sum_{k=3}^{\infty}\xi^{-k}w_{k-3}(x)\right).\end{aligned} \tag{32}$$

Here we find the "gauge" subgroup to be the (centreless) Virasoro group

$$g_1(L)=\exp[\phi_1(x)\xi]\,, \tag{33}$$

$$u_{-1}(F_{\phi_1}(x))=\partial_x F_{\phi_1}(x),\quad \text{with } F_{\phi_1}(x)\equiv\exp[\phi_1(x)\partial_x]x\,. \tag{34}$$

In (34) $F_{\phi_1}(x)$ denotes the global group Virasoro diffeomorphism generated by the Virasoro algebra element

$\phi_1(x)\xi \simeq \phi_1(x)\partial_x$. Note that all exponents involving symbols are operator ones. To obtain (32) we use the simple formulas

$$\exp[\phi_1(x)\xi]\xi\exp[-\phi_1(x)\xi] = \frac{1}{\partial_x F_{\phi_1}(x)}\xi, \quad \exp[\phi_1(x)\xi]u(x)\exp[-\phi_1(x)\xi] = u(F_{\phi_1}(x)) . \tag{35}$$

In the present case the analogue of (28) reads

$$\frac{\delta g_1(L)(\zeta, y)}{\delta L(\xi, x)} g_1^{-1}(L)(\zeta, y) = -\tfrac{1}{2}\epsilon(x - F_{\phi_1}(y)) \frac{1}{u_{-1}^2(x)} \xi^{-2}\xi. \tag{36}$$

Specializing formula (24) yields for $\tilde{L} \equiv \exp(\phi_1\xi)L\exp(-\phi_1\xi)$:

$$\begin{aligned}\{\langle\tilde{L}|\tilde{X}\rangle, \langle\tilde{L}|\tilde{Y}\rangle\}_{\mathrm{KP}_{l=2}} = &-\langle\tilde{L}|\exp(\phi_1\xi)\{[[\exp(-\phi_1\xi)\tilde{X}\exp(\phi_1\xi)]_{\geqslant 2}, [\exp(-\phi_1\xi)\tilde{Y}\exp(\phi_1\xi)]_{\geqslant 2}] \\ &-[(a_2(\phi_1, \tilde{L})(\tilde{X}))_{\leqslant 1}, (A_2(\phi_1, \tilde{L})(\tilde{Y}))_{\leqslant 1}]\}\exp(-\phi_1\xi)\rangle ,\end{aligned} \tag{37}$$

where $A_2(\phi_1, \tilde{L})(\tilde{X}) = \exp(-\phi_1\xi)\{\tilde{X} + (\partial_x^{-1}[\tilde{X}, \tilde{L}]_{(-2)})\xi^{-2}\}\exp(\phi_1\xi)$. Similarly to what happens in (29), in any term of the form $\langle\tilde{L}|\tilde{Z}\rangle$ on the RHS of (37), only the projection $\tilde{Z}_{\geqslant -2}$ contributes. The subscript (-2) means taking the coefficient in front of ξ^{-2} in the corresponding symbol expansion.

Noting that $\mathrm{Ad}(g_1^{-1}(L))$ preserves the splitting $\tilde{X} = \tilde{X}_{\geqslant 1} + \tilde{X}_{\leqslant 0}$ corresponding to $\mathrm{KP}_{l=1}$ (13), one can rewrite (37) as

$$\{\langle\tilde{L}(L)|\tilde{X}\rangle, \langle\tilde{L}(L)|\tilde{Y}\rangle\}_{\mathrm{KP}_{l=2}} = -\langle\tilde{L}(L)|[\tilde{X}_{\geqslant 1}, \tilde{Y}_{\geqslant 1}] - [\tilde{X}_{\leqslant 0}, \tilde{Y}_{\leqslant 0}]\rangle \tag{38}$$

$$+\langle\tilde{L}(L)|[\tilde{X} + (\partial_x^{-1}[\tilde{X}, \tilde{L}]_{(-2)})\xi^{-2}, \exp(\phi_1\xi)\{[\exp(-\phi_1\xi)\tilde{Y}\exp(\phi_1\xi)]_{(1)}\xi\}\exp(-\phi_1\xi)]\rangle \tag{39}$$

$$-\langle\tilde{L}(L)|[\tilde{Y} + (\partial_x^{-1}[\tilde{Y}, \tilde{L}]_{(-2)})\xi^{-2}, \exp(\phi_1\xi)\{[\exp(-\phi_1\xi)\tilde{X}\exp(\phi_1\xi)]_{(1)}\xi\}\exp(-\phi_1\xi)]\rangle . \tag{40}$$

Now, accounting for the structure of $\tilde{L}(L)$ (32), one can easily show that both terms (39) and (40) vanish separately. Thus, we are left with (38) only, i.e.,

$$\{\langle\tilde{L}(L)|\tilde{X}\rangle, \langle\tilde{L}(L)|\tilde{Y}\rangle\}_{\mathrm{KP}_{l=2}} = \{\langle\tilde{L}|\tilde{X}\rangle, \langle\tilde{L}|\tilde{Y}\rangle\}_{\mathrm{KP}_{l=1}} , \tag{41}$$

which establishes the "gauge" equivalence of $\mathrm{KP}_{l=2}$ and $\mathrm{KP}_{l=1}$, and because of (31), also the "gauge" equivalence of $\mathrm{KP}_{l=2}$ to ordinary KP.

4. Applications. A new four-boson representation of $W_{1+\infty}$

4.1. Lenard relations

As mentioned in section 2.1 above, Lenard relations shown below eq. (4) played an important role in establishing the bi-hamiltonian structure for the ordinary KP hierarchy. Here we comment on how the "gauge" equivalence between the various KP hierarchies carries the Lenard relations over to the modified KP hierarchies. First we note that $\mathrm{Tr}\,\tilde{L}^n = \mathrm{Tr}\,L^n$, with L being a Lax operator in the modified KP hierarchy (notations as in (25), (32)), follows from the "gauge" equivalence and ensures that the hamiltonians H_n remain identical for all KP_l (upon extending H_n as functions from orbits $\mathrm{O}(R)$ to the whole dual space $\mathscr{G}^*$). For simplicity we now discuss the case of $\mathrm{KP}_{l=1}$ with the Lax operator L related through $\tilde{L} \equiv \exp(\phi_0)L\exp(-\phi_0)$ to the Lax $\tilde{L}$ of usual KP hierarchy. One can easily show that the ordinary Lenard relations $\{H_n, \tilde{L}\}_2 = \{H_{n+1}, \tilde{L}\}_1$ translate now to the new Lenard relations

$$\{H_{n+1}, L\}_1 + [\{H_{n+1}, \phi_0\}_1, L] = \{H_n, L\}_2 + [\{H_n, \phi_0\}_2, L] . \tag{42}$$

Especially for the two-boson R_1-orbit $L_+ = \xi + u_0 + \xi^{-1} u_1$ we get the relations $\{H_n, u_i\}_2 = \{H_{n+1}, u_i\}_1$, with $i=0$, 1, reproducing the second bracket structure found in refs. [13,24].

4.2. New representations of W-algebras

Here we will use the symplectic "gauge" equivalence map to construct a new representation of the $W_{1+\infty}$ algebra. Let us first recall eq. (25) and solve for the coefficients of the Lax operator on the LHS in terms of the coefficients given on the RHS of this equation. One easily finds

$$\tilde{u}_{k+1} = u_1 P_k(u_0) + \sum_{n=2}^{k+1} \binom{k}{n-1} v_{n-2} P_{k+1-n}(u_0), \quad k \geqslant 0 , \tag{43}$$

where $P_k(u_0) \equiv (\partial + u_0)^k \cdot 1$, are the so-called Faá di Bruno polynomials and the fields on the RHS satisfy the LP bracket structure described in (20). As a corollary of the symplectic character of the "gauge" transformation, we conclude that $\tilde{u}_{k+1}$ (43) satisfy the Poisson-bracket $W_{1+\infty}$ algebra described by the form $\Omega^{(0)}$ from (18) (note that the index labelling $\tilde{u}_{k+1}$ is precisely equal to its conformal spin). Specifically, putting in (43) all v_i to zero, we recover the two-boson representation $\tilde{u}_{k+1} = u_1 P_k(u_0)$ of the $W_{1+\infty}$ algebra described in refs. [13,24] (see also ref. [25] for another related two-boson representation). The semiclassical limit is simply obtained by taking $P_k(u_0) \to u_0^k$ in (43) and yields the generators of the $w_{1+\infty}$ algebra.

Similar considerations applied to $\mathrm{KP}_{l=2}$ result in a new non-standard four-boson representation of the $W_{1+\infty}$ algebra. Indeed, performing a "gauge" transformation consisting of a composition of (25) and (32) on the four-boson R_2-orbit $L_+ = \xi u_{-1} + u_0 + \xi^{-1} u_1 + \xi^{-2} u_2$ with ϕ_1 as in (34) and $\partial_x \phi_0(x) = (u_0 + \partial u_{-1})(F_{\phi_1}(x))$, we obtain the following KP $\tilde{L}$ operator:

$$\tilde{L} = \exp(\phi_0) \exp(\phi_1 \xi)(\xi u_{-1} + u_0 + \xi^{-1} u_1 + \xi^{-2} u_2) \exp(-\phi_1 \xi) \exp(-\phi_0) = \xi + \sum_{k \geqslant 1}^{\infty} \xi^{-k} U_k , \tag{44}$$

$$U_{k+1} \equiv \hat{u}_1 P_k(\hat{u}_0) + \hat{u}_2 Q_{k-1}(\hat{u}_0, \hat{v}_0) , \tag{45}$$

using the notations

$$\hat{u}_0(x) \equiv (u_0 + \partial u_{-1})(F_{\phi_1}(x)), \quad \hat{v}_0(x) \equiv \partial_x \ln(\partial_x F_{\phi_1}) , \tag{46}$$

$$\hat{u}_1(x) \equiv \partial_x F_{\phi_1} u_1(F_{\phi_1}(x)), \quad \hat{u}_2(x) \equiv (\partial_x F_{\phi_1})^2 u_2(F_{\phi_1}(x)) , \tag{47}$$

$$Q_k(\hat{u}_0, \hat{v}_0) \equiv \sum_{s=0}^{k} (\partial_x + \hat{u}_0 + \hat{v}_0)^{k-s} (\partial_x + \hat{u}_0)^s \cdot 1 \quad (k \geqslant 0) . \tag{48}$$

Again, because of the symplectic property of the "gauge" transformation (44), the fields U_k (45) span a $W_{1+\infty}$ LP bracket algebra realized in terms of the four-boson fields (46) and (47). The poisson-bracket algebra of the latter,

$$\{\hat{u}_1(x), \hat{u}_0(y)\} = -\partial_x \delta(x-y) , \tag{49}$$

$$\{\hat{u}_2(x), \hat{v}_0(y)\} = -\hat{v}_0(x) \partial_x \delta(x-y) + \partial_x^2 \delta(x-y) , \tag{50}$$

$$\{\hat{u}_2(x), \hat{u}_2(y)\} = -2\hat{u}_2(x) \partial_x \delta(x-y) - \partial_x \hat{u}_2 \delta(x-y) , \tag{51}$$

the rest being zero, is a direct sum of the Heisenberg algebra of $(\tilde{u}_0, \tilde{u}_1)$ with the conformal algebra of spin-2 and non-primary spin-1 fields $(\tilde{u}_2, \tilde{v}_0)$. Let us point out that the deformation of the conformal algebra (50), (51) with $\{\tilde{v}_0(x), \tilde{v}_0(y)\} = -c\partial_x \delta(x-y)$ already appeared in the construction [24] of the two-boson representation of the second bracket structure of KP, while the fields $(\tilde{u}_0, \tilde{u}_1)$ comprise the usual two-boson content of the $W_{1+\infty}$ representation (the first term on the RHS of (43)). The above four-boson construction brings these two Bose structures together to yield a new representation (45) of the $W_{1+\infty}$ algebra.

It is an interesting problem to study the quantization of the integrable system corresponding to the four-boson realization of KP (quantization of the two-boson realization of KP has already been undertaken in ref. [26]).

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